Energy Transmission in the Forbidden Band Gap of a Nonlinear Chain

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(Received 11 February 2002; published 9 September 2002)

A nonlinear chain driven by one end may propagate energy in the forbidden band gap by means of nonlinear modes. For harmonic driving at a given frequency, the process occurs at a threshold amplitude by sudden large energy flow that we call *nonlinear supratransmission*. The bifurcation of energy transmission is demonstrated numerically and experimentally on the chain of coupled pendula (sine-Gordon and nonlinear Klein-Gordon equations) and sustained by an extremely simple theory.

DOI: 10.1103/PhysRevLett.89.134102 PACS numbers: 05.45.Yv

Introduction.—A nonlinear chain of oscillators has a few striking fundamental properties, the first of which, known as the Fermi-Pasta-Ulam (FPU) recurrence phenomenon [1], is the *spectral localization* of energy: injected initially in one given eigenmode energy does not eventually distribute among higher modes (as one would normally expect from a quasilinear approach). The FPU discovery has been at the origin of nonlinear studies all over the world and has led to the birth of the *soliton* concept [2], soon followed by the creation of the *inverse spectral transform* and the concept of *integrability* [3–5].

A nonlinear chain also shows up *spatial localization* of energy in the form of nonlinear coherent structures, the solitons [6]. This universal behavior is well understood in the concept of integrability: any localized bounded initial condition eventually evolves to a number of isolated solitons and a vanishing background of quasilinear radiation.

Nonlinear energy localization is now deeply studied in the context of discrete systems where *intrinsic nonlinear modes* (breathers for short) have been shown to be the fundamental basic objects [7,8], and have been experimentally observed in *Josephson ladders* [9]. Moreover, as an effect of nonintegrability, nonlinear modes do exchange energy and the spatial localization acquires novel quite interesting features [10].

The behavior of a nonlinear medium submitted to boundary data, as opposed to initial conditions, is also of fundamental interest. One famous instance is the project of data transmission in optical fibers in a nonlinear Kerr regime for which the basic model is the nonlinear Schrödinger equation [11]. Another instance is the nonlinear property of self-induced transparency of a two-level system submitted to high-energy incident (resonant) laser pulse [12,13]. In those two cases the concept of integrability has proved its efficiency as the physical boundary value problem maps to a well posed Cauchy (initial value) problem.

There, the nonlinear coherent structures emerge from sufficiently energetic localized input pulse. Another fundamental question is then the behavior of a medium submitted to continuous wave radiation. Such problems have been studied so far mainly through peturbation of integrable models by *external driving,* see, e.g., [14] for a review and [15] for interesting recent developments. But the more basic problem of scattering of continuous wave onto nonlinear media has not been studied much. Preliminary numerical results have been obtained recently in [16], where a linear monochromatic wave is scattered on a nonlinear medium. The nonlinearity allows then wave transmission under nonlinear modes generation, with a threshold which was not given a theoretical ground.

Those problems share a common basic question: the response of a nonlinear medium to periodic boundary data. We demonstrate here the existence of a bifurcation of wave transmission within a forbidden band gap (FBG) in a nonlinear chain forced (periodically) at one end. The related brutal energy flow through the medium is illustrated in Fig. 1 where the energy E penetrating the medium is plotted as a function of the amplitude A of the boundary driving at a frequency in the FBG (see below for details).

The mechanism of this bifurcation takes its origin in nonlinear modes generation by the periodic forcing and allows for energy injected at one end of a chain to penetrate a medium by an intrinsic nonlinear process. This is called *nonlinear supratransmission,* and it is illustrated

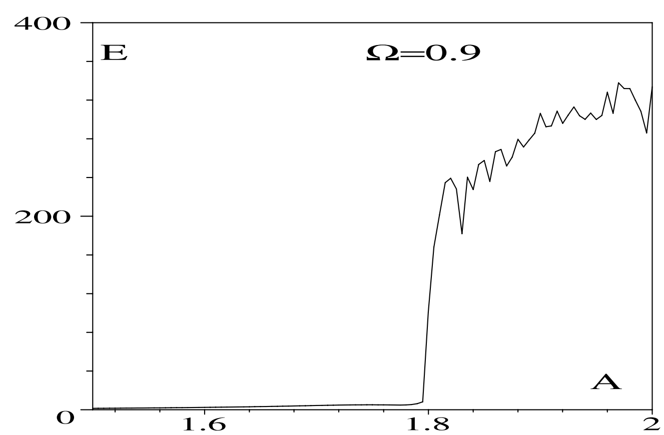

FIG. 1. Plot of the energy E injected in the sine-Gordon chain for $T = 140$ as a function of the driving amplitude A.

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both by numerical simulations of sine-Gordon and of nonlinear Klein-Gordon equations, and by experiments on a mechanical chain of pendula.

We shall first prove that, for sine-Gordon, the mechanism for nonlinear modes generation follows a very simple rule providing an explicit formula for the bifurcation diagram in the parameter space $\{A, \Omega\}$ where A is the amplitude of the driving and Ω its frequency. The rule states that energy penetrates the medium as soon as the amplitude A of the harmonic driving at frequency Ω exceeds the maximum amplitude of the static breather of the same frequency.

Nonlinear supratransmission holds also for the nonlinear Klein-Gordon chain obtained by the Taylor expansion of sine-Gordon. Then this process does not rely on integrability and is expected to be generic as soon as the model possesses a natural forbidden band gap.

We discover then that *harmonic phonon quenching* enhances nonlinear penetration: when the first significant harmonic generated by the nonlinearity (the third one in sine-Gordon) falls inside the FBG, the related phonons stick on the boundary and contribute to the driving allowing supratransmission at lower amplitude. This is an interesting property in view of experimental applications.

Last the energy transmission by means of nonlinear modes generation is explored, and the bifurcation is described by expressing the energy injected in the medium in terms of the driving amplitude, hence furnishing a striking view of the supratransmission process.

Model.—Consider the discrete sine-Gordon chain of coupled oscillators $u_n(t)$ (time is normalized to the eigenfrequency of the individual oscillator)

$$\ddot{u}_n - c^2(u_{n+1} - 2u_n + u_{n-1}) + \sin u_n = 0, \tag{1}$$

on a semi-infinite line $n > 0$ with a given initial-boundary value problem, namely, the data of the driving boundary $u_0(t)$, the initial positions $u_n(0)$, initial velocities $\dot{u}_n(0)$, and the boundary condition at the chain end. The linear dispersion relation $\omega(k)$ is given by

$$\omega^2 = 1 + 2c^2(1 - \cos k). \tag{2}$$

The chain will be submitted to external harmonic forcing $u_0(t) = A \sin\Omega t$ on a medium initially at rest. For a frequency Ω in the phonon band, quasilinear waves are generated in the medium and, for large enough amplitude, these waves will undergo Benjamin-Feir instability, hence creating localized excitations. These nonlinear modes have a very important role in the large time asymptotic properties of a nonlinear system and are suspected to be responsible for turbulentlike behavior [17].

We consider here a driving frequency in the FBG, namely, $\Omega < 1$, for which the linear theory would lead to the evanescent wave $A \sin(\Omega t) \exp[-\lambda n]$ with λ given by

$$\lambda = \operatorname{arccosh}\left(1 + \frac{1 - \Omega^2}{2c^2}\right). \tag{3}$$

In the nonlinear case, in order to fit the boundary condition (6), the medium *adjusts a static breather*

$$u_b(n, t) = 4 \arctan\left[\frac{\lambda c \sin(\Omega t)}{\Omega \cosh[\lambda(n + n_0)]}\right], \tag{4}$$

which is an exact solution in the continuous limit only but works well for strong coupling (the fully discrete case $c \ll 1$ will be considered separately).

Bifurcation process.—Adjusting a static breather actually means to adjust the value of the breather center $-n_0$ such that the oscillation amplitude at the boundary $n = 0$ matches the forcing amplitude. This works up to the maximum value A_s of the breather amplitude realized for $n_0 = 0$. Hence from (4), the threshold A_s reads as the following function of the frequency Ω:

$$A_s = 4 \arctan\left[\frac{c}{\Omega} \operatorname{arccosh}\left(1 + \frac{1 - \Omega^2}{2c^2}\right)\right], \tag{5}$$

which has the accurate simplified continuous approximation $4 \arctan[\sqrt{1 - \Omega^2}/\Omega]$.

This bifurcation threshold is now checked on numerical simulations of (1) with the following initial-boundary conditions:

$$u_0(t) = A \sin\Omega t, \qquad u_n(0) = 0, \qquad \dot{u}_n(0) = A\Omega e^{-\lambda n}. \tag{6}$$

The initial velocities are those of an evanescent wave such as to partly avoid the shock wave generated by vanishing initial velocities (the same results, but time consuming, are obtained for vanishing initial velocities and a driving amplitude smoothly growing from the value 0 to A). Finally, an infinite medium is simulated by an absorbing boundary. The simulations are made with the dsolve routine in MAPLE softward package with 10^5 maximum iterations. The absorbing end consists of adding a damping $\gamma \dot{u}_n$ in the model, with intensity γ slowly varying from 0 to 2 on the last 50 particles.

Results.—In order to generate a bifurcation diagram, one has to compare between simulations where the nonlinear supratransmission does or does not occur, as illustrated in Fig. 2 where the motion of one particle of the chain is plotted for driving amplitudes just below and just above the threshold. The simulation is performed with 200 particles with a coupling $c = 4$. Each large oscillation in the second figure corresponds to a breather (constituted of a single hump oscillating and propagating) passing by. Two of them are generated and cross the site 60 at times 120 and 160. The small oscillations seen between the humps are the harmonic phonons, mainly of frequency 3Ω.

We may now proceed with a systematic exploration of the chain response. The result is presented in Fig. 3 obtained for 200 particles with a coupling $c = 10$ (some experiments have actually been made with smaller coupling and fewer points to shorten computation times) for a typical time of 200 (for frequencies close to the gap value 1, time had to be increased up to 500). The points in Fig. 3

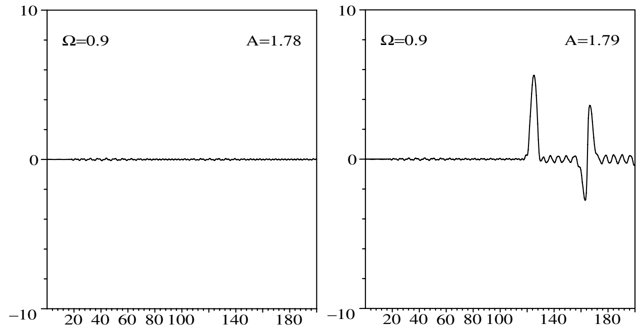

FIG. 2. Function $u_n(t)$ for $n = 60$ in the case $\Omega = 0.90$. The amplitudes are $A = 1.78 < A_s$ for the first figure and $A = 1.79 = A_s$ for the second.

are obtained with an absolute precision of 10^{-2} for the amplitude A. They are compared to the theoretical threshold expression (5) (continuous curve).

Harmonic phonon quenching.—Figure 3 shows excellent agreement with formula (5) at least in the region $0.34 < \Omega < 1$. Discrepencies are seen to occur starting below 0.33 and 0.18. This results from the driving which, thanks to the nonlinearity, generates phonons at multiple frequencies (here third and fifth). If these frequencies lie in the phonon band, the phonons move away from the boundary and have no effect on the forcing. If, however, they lie in the FBG, the related phonons do not propagate (which we call phonon quenching) and stick on the boundary where they add contribution to the driving.

This effect should thus disappear if the phonons are eliminated by driving the boundary with the exact breather expression (4) used to calculate $u_0(t)$, $u_n(0)$, and $\dot{u}_n(0)$. In that case we have checked that nonlinear supratransmission *never occurs* at an amplitude $A < A_s$, while it occurs for very small deformation of the perfect breather. For instance, by using $(1 + \epsilon)u_b(0, t)$ as the driving boundary, in the case $\Omega = 0.30$, supratransmission occurs for $\epsilon = 6 \times 10^{-4}$, i.e., for a driving amplitude $A = 5.0674$ instead of the threshold $A_s = 5.0644$.

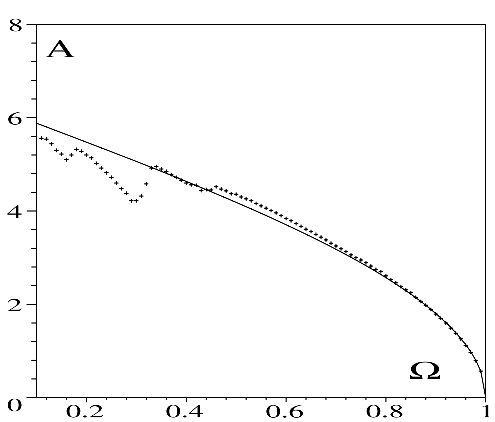

FIG. 3. Bifurcation diagram in the (A, Ω) plane. The curve is the graph of formula (5). The points indicate the lowest amplitude for which nonlinear supratransmission starts.

Energy transmission.—The nonlinear supratransmission allows energy to flow through the medium, and we compare here this energy flow below and above the threshold. The energy injected in the medium by the driving boundary is given at time T by

$$E = -c^2 \int_0^T dt \dot{u}_0(t)[u_1(t) - u_0(t)]. \tag{7}$$

In our case $u_0(t)$ is the driving (6) and the chain is supposed infinite with $u_n(t) \to 0$ as $n \to \infty$. Choosing for T an integer multiple of the period of excitation makes this energy vanish identically in the linear case if the driving frequency falls in the FBG.

In the nonlinear case, expression (7) is computed numerically. For a driving frequency 0.9 and amplitudes running from 1.5 to 2.0, we obtain Fig. 1 where the bifurcation is seen to occur for $A = 1.80$, the value predicted by formula (5). This simulation has been run for frequencies in the range $[0.2, 0.9]$, with expected results.

Nonlinear Klein-Gordon.—The approach stems from the existence of a breather solution of the model equation, allowing one to determine the threshold amplitude. Then a fundamental question is the role of *integrability* in the process of nonlinear supratransmission. To give an indication that this process is generic (with a stop gap), we have performed numerical simulations of the following nonlinear Klein-Gordon chain:

$$\ddot{u}_n - c^2(u_{n+1} - 2u_n + u_{n-1}) + u_n - \frac{1}{3!}u_n^3 + \frac{1}{5!}u_n^5 = 0, \tag{8}$$

the Taylor truncated expansion of sine-Gordon (the fifth order is kept to ensure a confining potential at large u_n).

Then this system is solved with the boundary driving (6) and the energy (7) is computed for the same parameter values as for Fig. 1. The result is displayed in Fig. 4. Nonlinear supratransmission is seen to still occur, though a nonlinear mode solution of the model does not exist. By scanning the frequency range in the gap, we have obtained that the process occurs down to $\Omega = 0.7$ and then disappears.

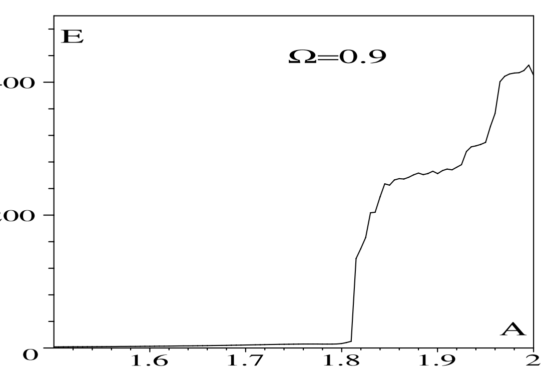

FIG. 4. Energy E injected in the Klein-Gordon nonlinear chain for $T = 140$ as a function of the driving amplitude A.

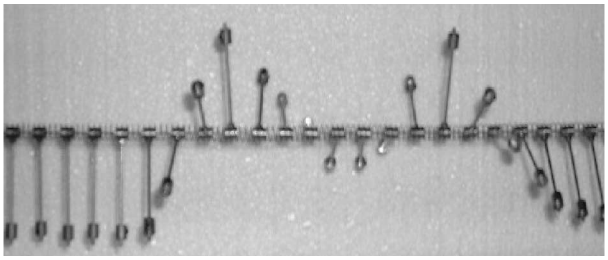

FIG. 5. Picture of a breather generated in a mechanical pendula chain driven at one end at a frequency in the forbidden band gap.

Experiment.—The phenomenon of nonlinear supratransmission can be experimentally realized on a mechanical pendula chain driven at one end by a periodic torque [18]. The detailed analysis of such experiments will be published later, but it is worth showing here a picture of a breather generated by the boundary driving at a frequency inside the FBG. The breather on Fig. 5 has been obtained with a chain of 48 pendula of angular eigenfrequency $\omega_0 = 15$ Hz (upper value of the FBG) by driving at (angular) frequency 12.7 Hz, which in the normalized units used here corresponds to $\Omega = 0.85$. The coupling constant has been measured to be $c = 32$.

Conclusion.—A novel fundamental property of a nonlinear medium has been unveiled, namely, the capacity to transmit energy under irradiation in a forbidden band gap by means of nonlinear mode generation. Theoretical construction of the bifurcation diagram is extremely simple when the one-breather solution is known and the numerical simulations fit strikingly well the theory. The generic mechanism of nonlinear mode generation is an instability which is expected to allow for prediction of the bifurcation even if the one-breather solution of the model is not known.

There remain, of course, many interesting open questions, and currently under study are the modulation of the driving signal, the effect of damping, viscosity, disorder, external bias, discreteness, etc. Another essential point is the search for nonlinear supratransmission in other physical situations. For instance, this result might provide understanding of the very mechanism of the *generation* of gap solitons in photonic band gap materials [19,20].

Last but not least, as the sine-Gordon chain is a model for discrete Josephson transmission lines [21], we expect interesting applications in this field. There, the boundary driving is realized by microwave irradiation through an antenna [22], and it actually corresponds to prescribing the *derivative* at the origin (Neuman condition). Preliminary numerical simulations has shown that nonlinear supratransmission also works and that the bifurcation process obeys the same type of simple rule. Detailed results on this question will be published later.

It is a pleasure to aknowledge enlightening discussions with M. J. Ablowitz, S. Aubry, G. Kopidakis, M. Schuster and A. V. Ustinov. This work has received the support of Contract No. INTAS 99-1782.

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